Accelerator-based Neutron Sources:

Past, Present, and Future

Joint Institute for Nuclear Research, DUBNA

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Topics

1912 Nuclear physics began.

1932 Chadwick's discovery of the neutron.

Early characterization of neutron-nuclear interactions.

Earliest accelerator-produced neutrons

1930's cyclotrons, 1950's MTA".

1950-'s Research reactor developments.

Small reactors--Atoms for Peace.

ING, A2R2, ILL, ANS,

1968 ANL Committee on Intense Neutron Sources.

New data, new ideas, new opportunities.

1973, ... ZING-P, ZING-P', IPNS, KENS, ISIS, SNS, JSNS, ESS

2000's Small reactor sources closing.

2000, ... The need for small, cheap neutron sources

1912: Cosmic Rays

The beginning of nuclear physics

A balloon ride that gave physics a lift

BAD SAAROW-PIESKOW, GERMANY

Grandson relates lessons from family member who discovered cosmic rays

BY BILL BREISKY

Precisely where Victor Hess, his electroscopes and his balloon touched down is a mystery.

What is known is that Hess and his crew of two landed shortly after noon, 100 years ago, near this little town in eastern Germany, and that a farmer brought the men to the Pieskow railroad station, where they took a train to Berlin and then another train home to Vienna.

The physicist Michael Walter says he found no reporting of the event in local newspapers at that time. But this week, there is something: an international symposium of physicists in Bad Saarow-Pieskow is being held to celebrate the centennial of the discovery of cosmic rays by Victor Francis Hess.

Dr. Walter, of the Institute for Theoretical Physics in Zurish will be an hand



International Herald Tribune 8 August, 2012

Victor Hess received the Nobel Prize in Physics in 1932.

1930s: Scientists used isotope sources to produce $Be(\alpha,n)$ neutrons

1932: James Chadwick discovered the neutron. He received the Nobel Prize in Physics in 1935.

James Chadwick's apparatus



Chadwick's Neutron Source

When a sheet of paraffin wax about 2 mm. thick was interposed in the path of the radiation just in front of the counter, the number of deflections recorded by the oscillograph increased markedly. This increase was due to particles

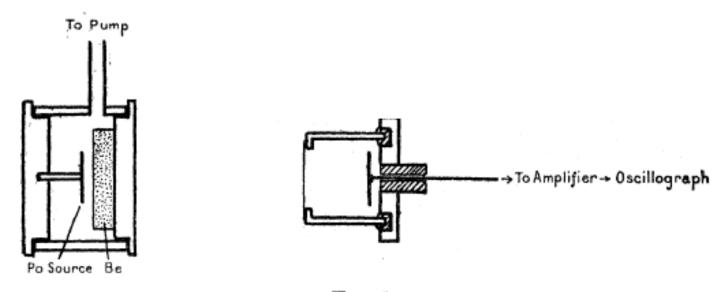


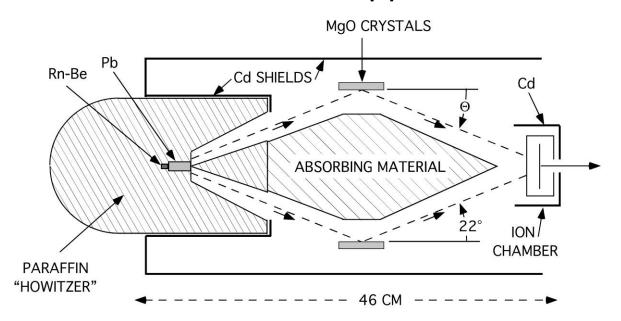
Fig. 1.

ejected from the paraffin wax so as to pass into the counter. By placing

1936: Neutrons as Waves

Mitchell and Powers and von Halban and Preiswerk demonstrate that neutrons behave as waves; they diffract from crystals as x-rays do.

Mitchell and Powers' apparatus



Cyclotrons

1930: Ernest Lawrence, U. C. Berkeley—cyclotron.

1930s: Cyclotrons evolve
Workers (Seaborg, ...) accelerate particles
to multi-MeV energies to produce isotopes.

Late 1930s: cyclotrons accelerate protons and deuterons to high enough energies to produce neutrons at rates greater than radioisotope sources, and to penetrate the coulomb barrier of heavy nuclei.

Properties of Neutrons

Mid-1930s: Enrico Fermi, University of Rome—both theorist and experimentalist, discovers many fundamental properties of neutrons, develops the theory of neutron-nuclear reactions.

Thermal neutrons—Fermi found that certain elements became more radioactive when exposed to neutrons in a water bath than from a bare source. He characterized elements in terms of their "aquaticity" and called such neutrons "thermal neutrons." Now we know the process as "Thermalization".

The Weizsäcker Mass Formula

1935: Carl von Weizsäcker, the semi-empirical nuclear mass formula.

A good accounting of the binding energy in terms of the numbers of neutrons and protons in nuclei.

Discovery of Fission

1938: Otto Hahn, Lise Meitner, and Fritz Strassmann, discover neutron-induced fission in uranium.

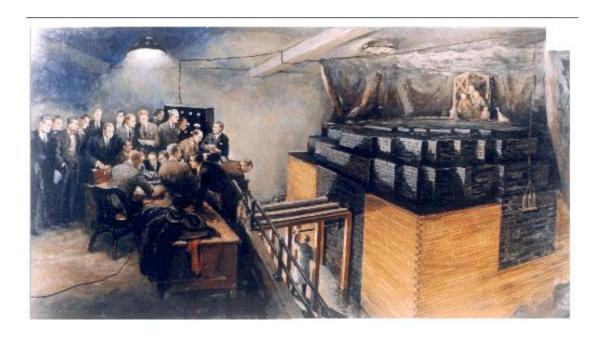
Immediately, people realized that excess neutrons emerge from that process. => reactors, bombs, Manhattan project.

Reactors

1942: Enrico Fermi and his team demonstrate the first self-sustaining nuclear reaction in the CP-1 reactor in Chicago.

Reactors rapidly evolved to become the most prolific neutron sources.

CP-1



1942: Forty-nine people attended the occasion on December 2, when the reactor went critical. Prominent were Enrico Fermi, Eugene Wigner, Leo Szilard, Walter Zinn, Herbert Anderson, Leona Marshall, Harold Agnew, Arthur Compton, Norman Hilberry, Frank Spedding,

1946: Walter Zinn at the diffractometer at CP-3

Beginning of neutron scattering studies of materials, ANL & ORNL



Those were the days when scientists wore suits and ties to work.

The Spallation-Fission Process

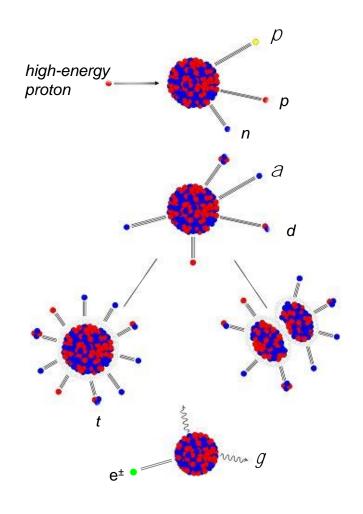
Schematic illustration of our modern understanding of the spallationfission (when fission is possible) process.

First stage: intranuclear cascade

Intermediate stage: preequilibrium

Second stage: evaporation and/or fission

Final stage: residual deexcitation



(Courtesy L. Waters, LANL)

Atmospheric Neutrons

Gradually, people became aware of neutrons produced in the atmosphere by energetic cosmic-ray protons (~ 10 GeV) in the spallation reaction.

Harold Agnew's 1944 Flying Neutron Detector (B-29)



Atmospheric Spallation Neutrons

Fermi, University of Chicago 1948 lectures—cosmic-ray-proton-induced neutron flux as a function of atmospheric Ch. X depth.

There are always neutrons around us. The thermal neutron flux at the Earth's surface is $\sim 10^{-4}-10^{-3}$ n/cm²-s, varying with atmospheric pressure (i.e., weather), tides, &C.

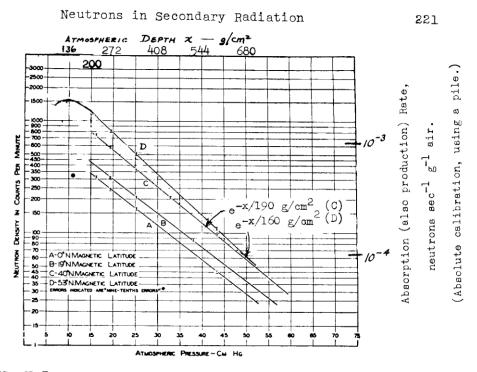


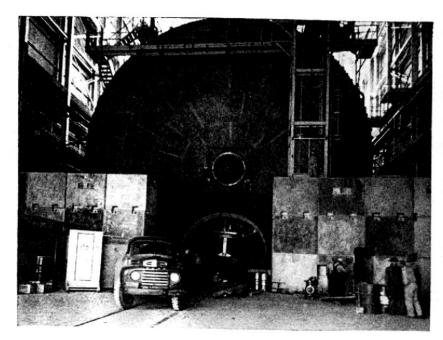
FIG. X.3 Apparent absorption of neutron-producing radiation at various magnetic latitudes*. See text, p. 220.

Accelerator-Produced Neutrons

1955: The MTA Linac. Eventually, workers tried out high-power accelerator-based neutron-producing facilities.

The accelerating cavities were very large because available high-power klystrons operated at only 12 MHz.

(Now commonly 800 MHz)



BIGGEST VACUUM VES-SEL ever built, 60 feet in diameter and 87 feet long, housed Materials Testing Accelerator. Men and truck in photo show comparative size. Door 20 feet in diameter, opened at tank's bottom, admitted rail car handling drift tubes. Concrete blocks like those seen formed unbroken wall of shielding while the accelerator was running.

Neutron Scattering

- Manhattan Project reactors—aimed to produce bomb fuel, ²³⁹Pu and to develop the relevant nuclear data.
- The involved scientists, Walter Zinn, Enrico Fermi, Ernest Wollan, Clifford Shull, and others, interested to apply their neutron beams for scientific purposes—fundamental physics experiments and materials science measurements revealed the uses of neutrons in science. Project managers allowed this without serious reservations.
- 1946: Many results were revealed in when classification of data was relaxed. These revelations led eventually to reactors designed to produce neutron beams for thermal-neutron scattering applications.

What Kind of Reactors?

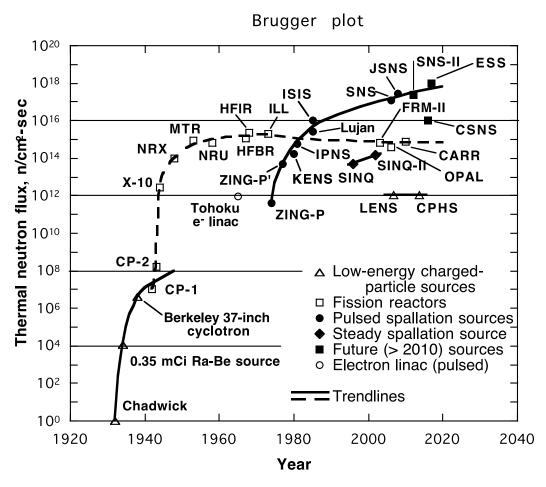
Why not use power reactors to produce neutron beams for research?"

Power reactors, high power—big and low-flux.

Research reactors, high flux— <u>small and dense</u>, challenging heat transfer limitations; <u>beam holes</u>, a "no-no" for power reactors.

Neutron Science Facilities

Reactors and Accelerator-Based Sources



A^2R^2

1960s: Argonne National Laboratory, proposal to build a high-flux research reactor, Argonne Advanced Research Reactor, A²R². The 5-MW CP-5 research reactor had been operating since 1954, supporting numerous neutron-related research programs.

1967: construction begins; I join a committee to define neutron-scattering instrumentation.

Committee meets once in early 1968.

April 1968: project canceled.

CINS Formed

1968: Committee on Intense Neutron Sources, CINS, to identify the best route for a facility to carry on neutron research at ANL.

Members

Argonne:

- T. V. Banfield, T. H. Blewitt, L. M. Bollinger (Chair after Oct '68),
- D. W. Connor P. R. Fields, M. Levinson (Chair before Oct '68),
- S. W. Peterson, G R. Ringo, A. B. Smith, R. L. Martin (later)

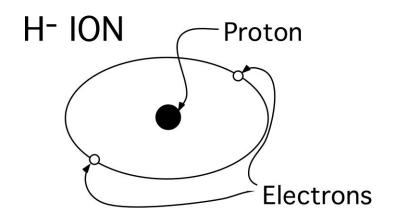
Universities:

J. M. Carpenter (U Mich), H. Danner (U Missouri), D. Glower, (Ohio State U), J. S. King (U Mich), L. H. Schwartz (Northwestern U), L. C. Teng (National Accelerator Laboratory)

CINS

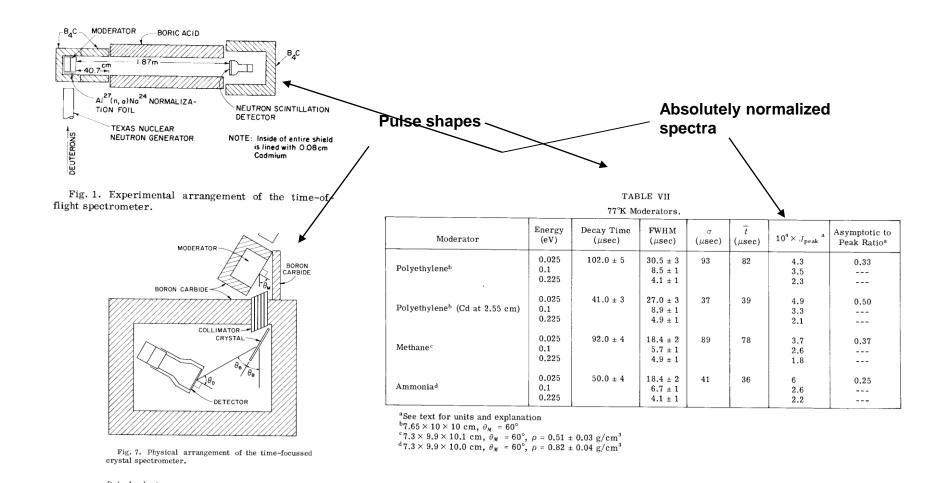
New information emerged—bright, negative hydrogen-ion (H⁻) sources (Dimov developed in Russia, brought back by Ron Martin), stripping injection and high-current proton synchrotrons.

These ideas were natural to Argonne, host to the 12-GeV Zero Gradient Synchrotron, ZGS.



CINS Data

1967-70: Kingsley Graham's thesis research,—absolutely-normalized data on performance of neutron moderators for pulsed neutron sources.



CINS: New Spallation-yield Data

1965: Fraser, et al., Data on spallation neutron production.

Measured for the Canadian Intense Neutron Generator project, ING (no MCNP then).

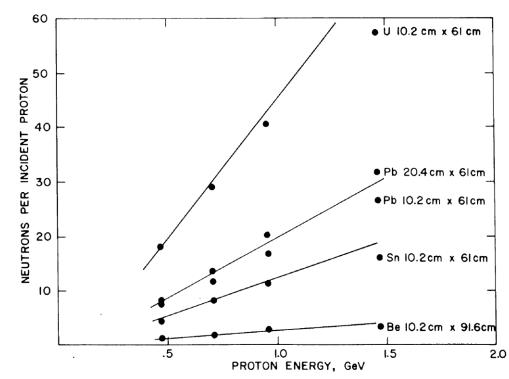


Fig. 2. Measured neutron yield vs proton energy for various targets³).

For 0.2 < E < 1.5 GeV, for A > 9 except Uranium,

 $Y = 0.1 (A + 20)(E_{GeV} - 0.12) n/p;$

 $Y = 50(E_{GeV} - 0.12)$ n/p for U-238.

Why Pulsed Spallation Sources?

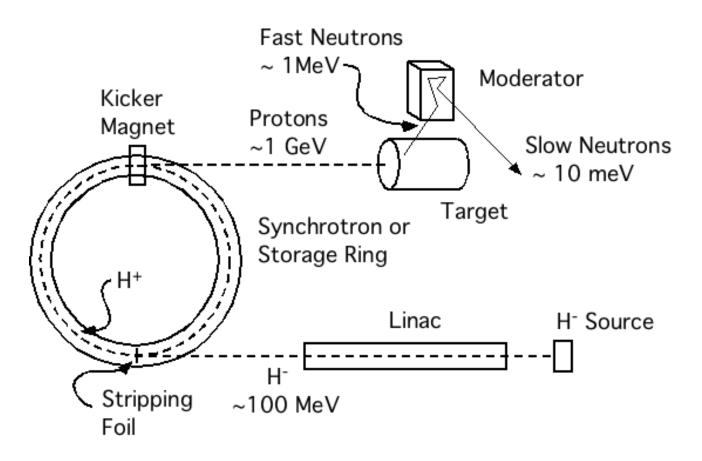
Spallation process—efficient: ~ 30 MeV heat per neutron, vs. ~ 200 MeV heat per useful neutron from fission.

Pulsed operation—high instantaneous power; long times between pulses for heat removal; favors cold moderators.

Pulsed sources—well suited to time-of-flight instrumentation; define the time origin.

Pulsed-source moderators—short neutron pulses; \sim constant $\Delta t/t$; high fluxes of epithermal neutrons.

Accelerator-Based Pulsed Neutron Source



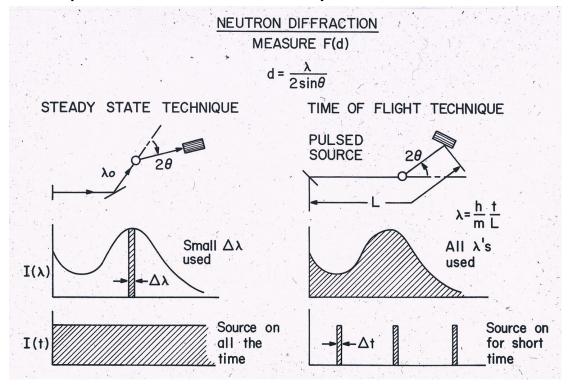
Moderator(s) close to the target slow down fast neutrons to energies useful for applications.

Steady vs. Pulsed Operation

Steady sources use *some* of the neutrons all of the time; pulsed sources use all of the neutrons some of the time.

Pulsed sources—accelerator delivers energy to the target in short (~ microsecond) bursts.

Heat is removed during the long interval between pulses.



Duty-Cycle Advantage

Pulsed sources relate naturally to accelerators, most types of which intrinsically operate in pulsed mode. In this mode, pulsed sources have a <u>duty-cycle advantage</u>—the source is *on* and at full power only part of the time and *off* most of the time, when heat in the target and moderators is (slowly) removed.

If the source is *on* for only time Δt_{source} and pulses at frequency f, the peak power is related to the average power as

$$P_{\text{peak}} = P_{\text{average}} / f \Delta t_{\text{source}}$$
.

Duty-Cycle Advantage cont'd

The same is true for the moderated neutron flux: the source is on for the duration of the pulse, which depends on the wavelength.

For example, for f=20 Hz and $\Delta t_{mod}(\lambda)=50$ μs , the duty-cycle factor is $1/f\Delta t_{mod}$ and the peak flux is $\phi_{peak} = \phi_{ave}/f\Delta t_{mod} = 10^3 \, \phi_{ave}!$

1969: Recommendations from CINS

Increase ZGS power—raise the injection energy, 50-MeV from the linac, to 500 MeV to increase the synchrotron space charge limit, using a Booster synchrotron with H⁻ stripping injection. (ZGS would use the Booster only about 10% of the time.)

Build a pulsed spallation neutron source, using the Booster accelerator part time.

The Booster Accelerator and ZING

The expected performance of the Booster, the absolutelynormalized ING neutron yield data and Graham's absolutely-normalized moderator data provided the basis for evaluating the spallation source concept.

We called it the ZGS Intense Neutron Generator, ZING. George Summerfield, my close friend, called it the Zero Intensity Neutron Generator. (~99.99% true)

I estimated promising performance: Promising, but a little short of what might be desired. Questions remained about the effectiveness of neutron-scattering instruments.

Post-CINS Activities

1970: Argonne adapts a decommissioned 2-GeV electron synchrotron from Cornell University to use as a 200-MeV prototype proton machine for testing H-injection, Booster-1, for developing the 500-MeV Booster-2 for ZGS.

1971-72: I spent a sabbatical year at ANL, partly doing neutron scattering experiments at CP-5 and partly working on the ZING idea.

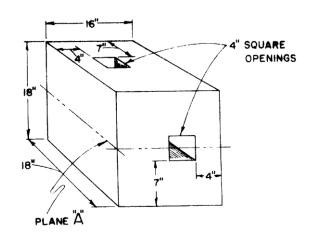
1972: The ZING Mockup

Argonne allowed me to build a radioisotope-driven model of the ZING target-moderator system, the ZING Mockup.

I used left-over A²R² Be blocks and learned track-etch neutron detection.

The idea was for a decoupled beryllium reflector to increase the intensity of moderated neutrons, which worked well.

I patented the idea (\$26) and returned to Michigan.



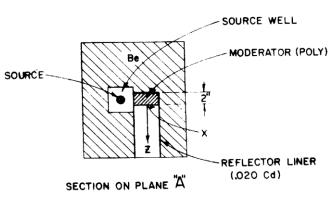


Figure XIV Mockup of the ZING source, moderator and reflector, with a beam port. The assembly was surrounded by 6" Benelex.

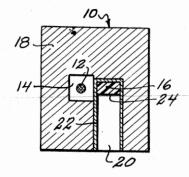
1973: ZING Patent

This was the first revelation of the decoupled beryllium moderatorreflector concept, now central to all modern pulsed neutron sources.

United States Patent 1191 3,778,627 Carpenter [45] Dec. 11, 1973 [54] HIGH INTENSITY, PULSED THERMAL 3,345,515 10/1967 [75] Inventor: John M. Carpenter, Ann Arbor, Primary Examiner-James W. Lawrence Assistant Examiner-T. N. Grigsby [73] Assignee: The United States of America as Attorney-Roland A. Anderson represented by the United States Atomic Energy Commission, Washington, D.C. This invention relates to a high intensity, pulsed ther-[22] Filed: Apr. 17, 1973 mal neutron source comprising a neutron-producing source which emits pulses of fast neutrons, a modera-[21] Appl. No.: 351,893 tor block adjacent to the fast neutron source, a reflector block which encases the fast neutron source and [52] U.S. Cl. the moderator block and has a thermal neutron exit Int. Cl. port extending therethrough from the moderator .. G21g 3/04 Field of Search 250/499, 500, 501, block, and a neutron energy-dependent decoupling reflector liner covering the interior surfaces of the thermal neutron exit port and surrounding all surfaces of References Cited the moderator block except the surface viewed by the thermal neutron exit port. UNITED STATES PATENTS

14 Claims, 4 Drawing Figures

8/1941 Kallmann



Continued Activity

1973: Argonne workshop. Sam Werner and I convened about 30 people to evaluate the ZING idea.

Motoharu Kimura came, known for his work on the electron-linac pulsed neutron source at Tohoku University—"You must build a prototype. I will help."

Summer, 1973: ZING-P,—Kimura returned to work on the design of the prototype. I took a semester leave of absence from Michigan, working with Argonne engineer Bob Kleb and Kimura and his protégé Noboru Watanabe.

Obtained \$30,000. from Argonne, built ZING-P, completed and first operated in January 1974.

It was the first of its kind. It worked!

1973 Workshop

By 1973, ideas were sufficiently along to encourage a workshop to evaluate ZING. David Price and I convened 36 scientists from Argonne, Los Alamos, ORNL, MTR (Idaho), Hanford, ILL, RAL, Tohoku U, from 29 April to 4 May, 1973 at Argonne, to address APPLICATIONS OF A PULSED SPALLATION NEUTRON SOURCE, based on the ZING concept.

The workshop report is ANL-8032.

ZING-P

The 1973 workshop concluded that an accelerator-based pulsed spallation source is a good idea. Motoharu Kimura, famous in Japan for establishing the Tohoku (e⁻) linac neutron source, at first skeptical, then convinced, declared to me "You must build a prototype."

No such thing had existed. Kimura said, "I will help." He returned to Japan and soon returned with his protege' Noboru Watanabe. I arranged a 6-month leave from Michigan. David Price assigned Robert Kleb to work with us. With Kurt Skold, we designed the prototype, ZING-P, to operate on the proton beam from Booster-1, and two scattering instruments on vertical neutron beams.

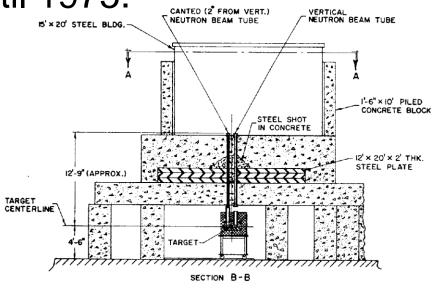
Moto Kimura observing the landscape



The ZING Prototype ZING-P

October, 1973: Bob Kleb, Kimura, and I worked out the details of ZING-P. The target was ½ of a lead brick. ANL allocated \$30,000 and assigned people for the job. Tom Banfield managed the project. Completed in January, 1974. ZING-P operated

until 1975.



BORON CARBDE & LEAD SHOT

BORON CARBDE

LEAD SHIELDING

POLYETHYLENE
MODERATOR

LEAD TARGET

MODERATOR

DRAWER

Fig. 2. Cross section through the ZING Prototype target-moderator

Fig. 2. Cross section through the ZING Prototype target-moderator assembly. Lead shielding bricks 5.1 x 10.2 x 20.3 cm give scale to the figure.

Fig. 1. Cross section through the ZING Prototype facility. Protons strike the target from behind this view.

First publication

1975: We published the results of the first pulsed spallation neutron source science measurements in the proceedings of the Conference on Neutron Diffraction, Petten, the Netherlands, August 1975,

"Neutron Diffraction Measurements on Powder Samples Using The ZING-P Pulsed Neutron Source at Argonne."

- J. M. Carpenter, M. H. Mueller, R A. Beyerlein, T. G. Worlton, J. D. Jorgensen, T. O. Brun, K. Skoeld, C. A. Pelizzari, S. W. Peterson, N. Watanabe, M. Kimura, and J. E. Gunning*.
- *First PhD thesis (UMich) from a pulsed spallation source.

Charles Pelizzari at ZING-P at 391-B



Why I left the University of Michigan and go to Argonne

ZING-P—a successful demonstration, operated from 1974 until 1975. I needed to be at Argonne full time to guide the continuing program.

ZING-P'—In early 1975 the Argonne team floated a proposal to the AEC to build the full-scale, larger ZING machine, based on the 500-MeV Booster-2, and also designed and built a more powerful prototype, ZING-P', to be operated in the meantime.

ZING-P' ran between 1977 and 1980.

1975 Workshop

By 1975 ZING-P had proved out expectations and shut down for installation of the 500-MeV Booster-2. Sam Werner, my UM colleague, and I convened another workshop at Argonne, on USES OF ADVANCED PULSED NEUTRON SOURCES, 21-24 October, 1975. About 70 people attended from world-wide institutions, working together in subgroups on 9 topics. The report, ANL-76-10, Volume 2, was issued. (There was no Volume 1.)

The participants endorsed the pulsed spallation source idea.

IPNS

By that time Paul McDaniel, then head of Argonne Universities Association, convinced me that we could not name our concept ZING, "too cute", he said. He recalled his experience at a hearing for an earlier Argonne research reactor proposal called "Mighty Mouse." "No, Suh," said the Senator, and that was that. Paul recommended "Call it something that you can't pronounce."

I decided to call it the Intense Pulsed Neutron Source, IPNS, which stuck from then on.

ZING and IPNS

1975: AEC rejected the ZING proposal—"not sufficiently ambitious."

The ANL team conceived a much more powerful version— an 800-MeV synchrotron delivering 400 kW beam power, the Intense Pulsed Neutron Source, IPNS, to be built in two phases. The proposal, ANL-78-88, was issued in 1977.

1977: AEC rejected the 400-kW IPNS proposal—it was "too big."

1981: Eventually, with AEC (then ERDA, precursor to DOE) support, we completed ZING, now designated IPNS-1, which ran until 2008, with 12 neutron scattering instruments, based on the 500 MeV Booster 2.

We built IPNS-1 and its prototypes using recycled and abandoned hardware, buildings, and infrastructure. They all operated as user facilities, serving the scientific community. And all the people involved worked very hard, together, and had fun.

1977: ICANS

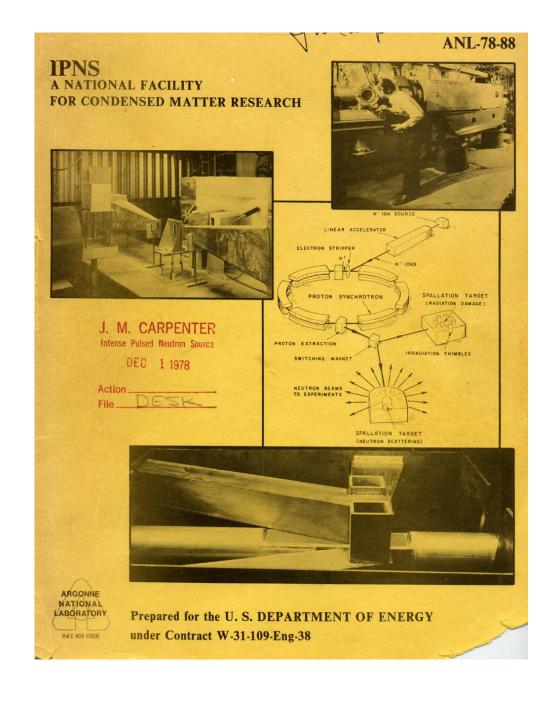
R. G. Fluharty (LANL), G. C. Sterling and L. C. W. Hobbis (Rutherford Lab UK), M. Kimura (Tohoku U) and I (ANL), meeting at Argonne, discussed the need for a forum where those working on pulsed spallation neutron sources could meet to discuss ideas and initiatives, to share experiences and to organize collaborations.

We called it the INTERNATIONAL COLLABORATION ON ADVANCED NEUTRON SOURCES, ICANS. The 22nd meeting, ICANS XXI, took place in Oxford, UK, in March, 2017. The next meeting, ICANS XXII, will be in Chattanooga, TN, in 1019.

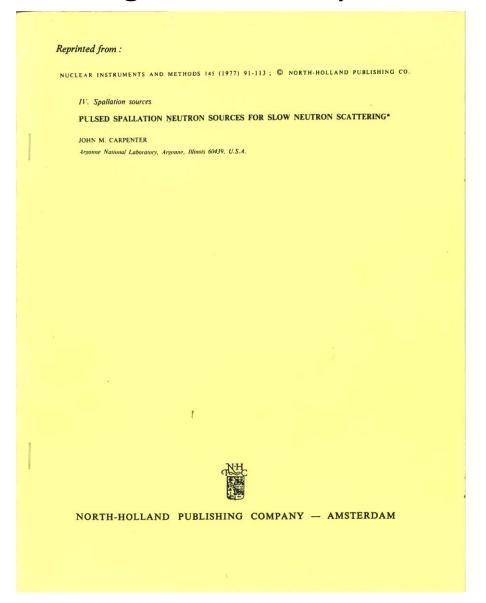
Now, about 10 labs, 200-300 participants, ~20,000 pages of proceedings.

The proposal for IPNS, a 400-kW Intense Pulsed Neutron Source. Two targets, same accelerator. Two phases, IPNS-1, 2.

We shared these ideas widely. This report was the basis for the ISIS facility in UK, which was completed in 1985, and soon demonstrated scientific effectiveness comparable to that of existing research reactors.



1977: Pulsed Spallation Neutron Sources for Slow Neutron Scattering, NIM 145, p 91-113.

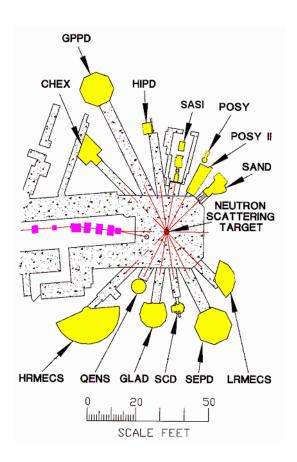


ICANS details

ICANS was formed as a loose consortium of institutions with responsibilities to host meetings and publish proceedings. At each meeting, the host would bear meeting expenses, participants would pay their own expenses, and the need, venue, topics and dates for the next meeting would be agreed. Each member would identify a "contact" who would promote the affairs in his/her laboratory. The host of the next meeting would be the leader of the consortium. We envisioned yearly meetings. Groups wishing to join would communicate with the current leader, who would solicit agreement from members.

IPNS 1982-2008





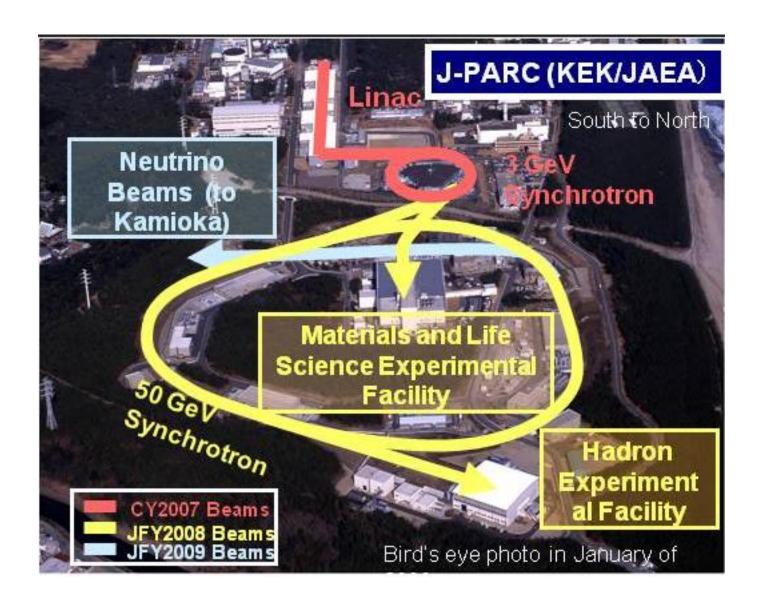
ISIS (RAL UK) 1985-present



SNS in 2012 (ORNL 2006-present)



J-PARC in January 2011 (2007-present)



ESS, The European Spallation Source at Lund, Sweden in 2016 (start 2019)



CSNS Guangdong, China (startup ~ 2020) No picture available

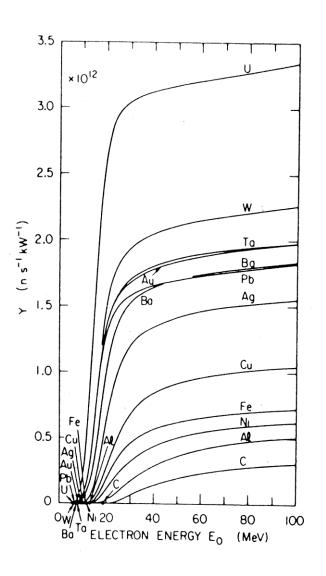
The need for small neutron sources

Small reactor sources are closing.

1-10 MW, Atoms-for-Peace are old, outmoded, sometimes neglected. Includes some not so small (e.g. ORPHEE).

Capacity for technical developments, training, easy test experiments, isotope production, therapy, is being lost.

Need small, cheap neutron sources: Low-energy reactions, electron-bremsstrahlung-photoneutrons, repurposing of abandoned accelerators, exotic schemes, ... ?



Electron linac e- bremsstrahlung photoneutron sources

Heavy element targets preferred.

For W on the plateau, the energy deposited in the target per neutron produced is

 $E/Y(E) \gg 2800 \text{ MeV/ neutron}$

Low-Energy Neutron Sources

Advantages—

Low cost of accelerator and operation.

Minimal shielding.

Cold moderators easy.

Easily adaptable for testing, development and training.

Disadvantages

Modest flux implies long experiment times.

Only a few neutron beams.

Low-Energy Neutron Sources (Driven by low-energy accelerators)

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For example,

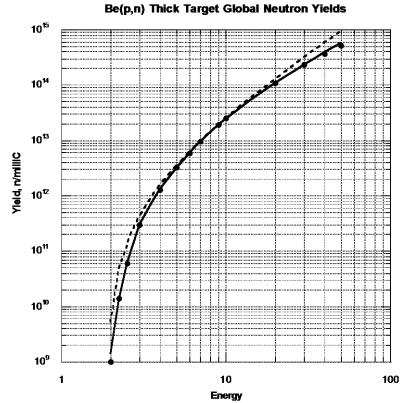
p + {}^{9}Be —> 2\alpha +n + p'

~1300 MeV/n @ E_p = 13 MeV

(heat deposited in ~ 1.1 mm)

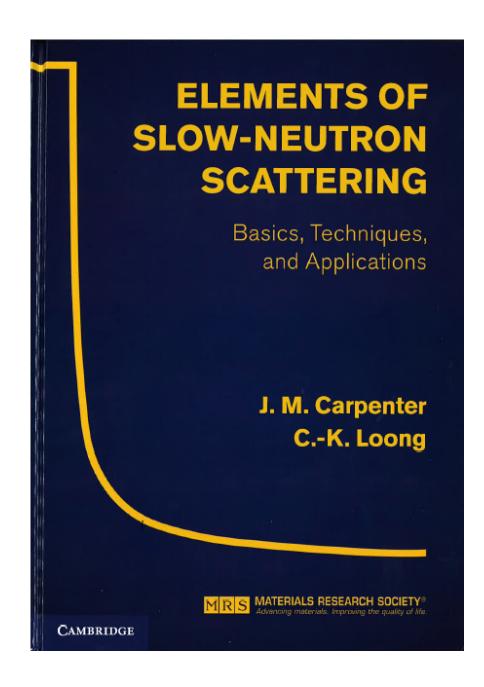
~ 3.5x10<sup>-3</sup> n/p
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Be (p,n) Neutron Yields



A simple function fits the Be(p,n) data reasonably well: dashed line, $Y(E_p) = 3.42 \times 10^8 (E_p - 1.87_{MeV})^{2.0}$ neutrons per millicoulomb.

Recent book ISBN 978-0-521-85781-9



Thank You!

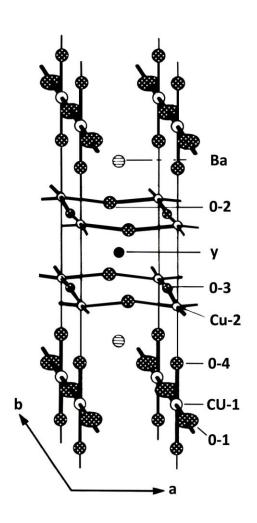
Applications of Neutron Scattering

Perhaps the main application of the big neutron sources, reactors, and accelerator-based sources is slow-neutron scattering. Other applications, including fundamental physics, materials testing, materials irradiation, and activation analysis. But a natural one for Nukes, like me, is slow-neutron scattering as a probe of materials structure and atomic motions. The community of users world wide numbers in many thousands. Following are some examples of work done with slow-neutron scattering.

Structures of high-temperature superconductors

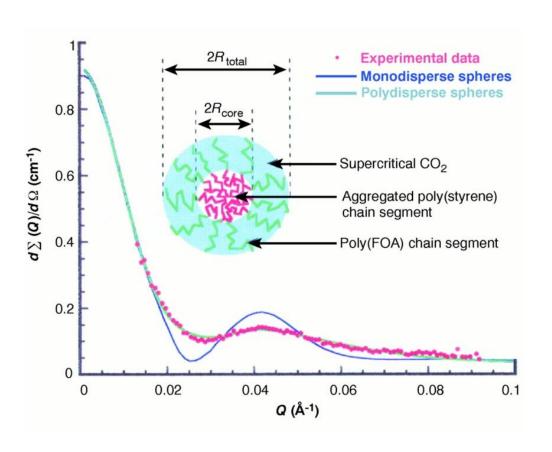
Rietveldt refinement of the data produced what is now the well-known structure. Within days workers at KENS and ISIS obtained identical results.

Pulsed-source powder diffractometers remain the method of choice for determining the structures of newly developed hi-T_c materials.



SANS research team wins Presidential Green Chemistry award for a CO₂ surfactant

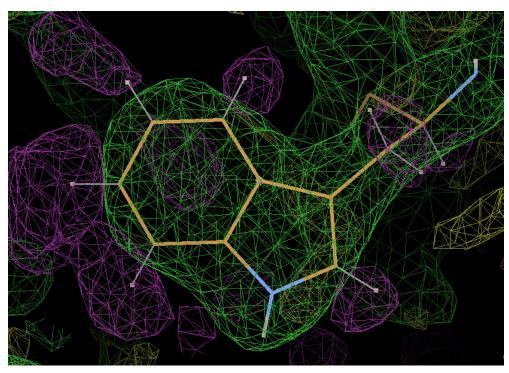
Scattering function of surfactant on polymer droplet in CO₂



Surfactants (long molecules, one end of which likes CO₂ and the other end of which likes hydrocarbon) would dissolve in supercritical CO₂ and form micelles. Scientists showed that their surfactant can suspend up to 20 wt% of hydrocarbon in CO₂. The figure shows the results of SANS measurements.

Neutron coherent scattering length density contour map

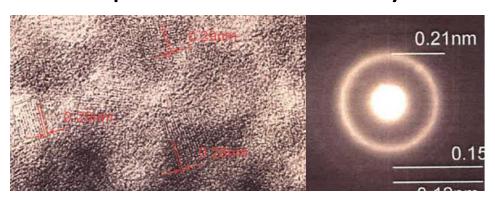
lysozyme and rubridoxin



Purple, hydrogen (negative scattering length)
Green, carbon, oxygen, etc. (positive scattering lengths)

Local structure of MgCoH

New Mg-Co based hydrogen storage material. Amorphous substance synthesized by ball milling.



5 nm crystals & amorphous part.

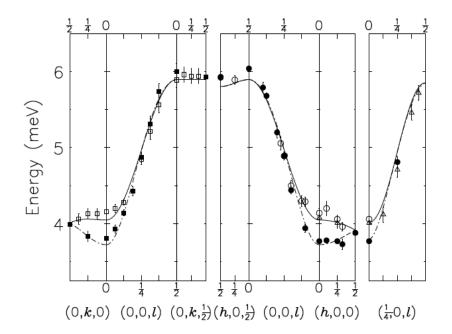
Conventional methods cannot analyze this *complicated* structure. No total scattering structure solution tools exist.

~ 2.5 wt% of hydrogen absorbed at low temp.

PDF method based on both x-ray and neutron diffraction necessary to understand the structure.

Triple-axis spectrometer

The magnon dispersion relations of CoCl₂•2D₂O at 7 K.





The HB1 spectrometer at HFIR

High resolution spectroscopy

Disorder effects in rotational tunneling in CH₄:CH₂D₂(7.3%)

Backscattering spectrometer BASIS at SNS:

E-Resolution ~ 3 micro-eV D/
$$I/ = \cot(q_s/2) Dq_s/2$$

W. Press, I. Krasnow, et al.

